# TWO TENTATIVE SECTIONS OF MATHEMATICS FOR ECONOMICS

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## I

This is not written from theoretical standpoint but from pedagogical. It is over one century and a quarter since A. Cournot employed mathematics for economic problems in his monumental *Recherches sur les principes mathématiques de la théorie des richesses.* It is already twenty years since P. A. Samuelson formulated and systematized economic theories mathematically in his famous *Foundations of Economic Analysis.* At present a great many books are existing on mathematics for economics and mathematical economics. In preparing and giving lectures on mathematics for economics, however, I know that there are few books appropriate for and recommendable to students.

Some books contain no examples or too few, if any, from economics though they are entitled *Mathematics for Economics.* Some books give pretty many examples, but they are often independent one another and there is no systematic connection among them. Thus those students who try to make use of mathematics in economic studies are disappointed that they can not learn how to formulate economic problems mathematically and systematize their results of research.

I think, therefore, this kind of books should tell not only how to formulate economic concepts and problems in terms of mathematical symbols but also how to deduce relevant conclusions by means of mathematical techniques.

Some books introduce too high grade of simplification in theories e.g. constancy of cost of production and linearity of economic functions. Thus those students who are interested in reality may feel that application of mathematics in economics is not very realistic.

I might say, therefore, that simplification should be avoided as far as possible.

I intend to produce two tentative sections of mathematics for economics herein.

## II

**Two Different Points of View in Monopoly** A monopolist firm produces goods $X$ under known condition represented by the total cost function

$$c=C(x) \qquad (1),$$

where $c$ is the total cost for output $x$. The firm fixes $x$ and leaves the price $p$ determined by the demand condition

$$p=P(x) \qquad (2).$$

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Thus the net revenue $\pi$ of the firm is a function

$$\pi=\Pi(x)\equiv xP(x)-C(x) \tag{3}$$

of $x$, and the firm is assumed to act so as to maximize its net revenue.

Differentiating (3) with respect to $x$, therefore, we have the equation

$$\frac{d\pi}{dx}\equiv P(x)+x\frac{dP}{dx}-\frac{dC}{dx}=0 \tag{4}$$

to be solved for $x$. And $x$ must satisfy a second condition

$$\frac{d^2\pi}{dx^2}\equiv 2\frac{dP}{dx}+x\frac{d^2P}{dx^2}-\frac{d^2C}{dx^2}<0 \tag{5}$$

for maximal net revenue as opposed to minimal.

There is, however, another point of view to be taken: the firm fixes price $p$ and the demand $x$ at $p$ is left to be determined by the inverse function

$$x=X(p) \tag{6}$$

of (2).

From this viewpoint, the net revenue $\pi$ is a function

$$\pi=pX(p)-C(X(p))$$

of $p$ and the equilibrium and stability conditions satisfied by $p$ are

$$\frac{d\pi}{dp}=0 \quad \text{and} \quad \frac{d^2\pi}{dp^2}<0 \tag{7}$$

respectively.

Differentiation of (3) with respect to $x$ and that of (6) with respect to $p$ yield

$$\frac{d\pi}{dp}=\frac{d\pi}{dx}\cdot\frac{dx}{dp} \tag{8}$$

Since $x$ is a decreasing function of $p$ by (6),

$$\frac{dx}{dp}<0. \tag{9}$$

Combination of (8) with (9) reveals to us that (4) and $(7)_1$ are equivalent to each other.

(8) differentiated with respect to $p$ yields

$$\frac{d^2\pi}{dp^2}=\frac{d^2\pi}{dx^2}\left(\frac{dx}{dp}\right)^2+\frac{d\pi}{dx}\cdot\frac{d^2x}{dp^2}$$

whose second term vanishes due to (4), and we have the equality

$$\frac{d^2\pi}{dp^2}=\frac{d^2\pi}{dx^2}\left(\frac{dx}{dp}\right)^2.$$

From this equality and the inequality

$$\left(\frac{dx}{dp}\right)^2>0$$

we infer that (5) and $(7)_2$ are equivalent to each other.

The respective equivalences of (4) to $(7)_1$ and (5) to $(7)_2$ conclude that two different analyses of the problem lead to the identical result.

**Taxation and Monopoly** Suppose that the firm is subject to taxes. Then its net revenue becomes

$$\pi_T=\Pi_T(x,t)\equiv xP(x)-C(x)-T(x,t),$$

where $T(x,t)$ is the tax amount depending on $x$ and the taxation parameter $t$ with

$$\frac{\partial T}{\partial x}>0, \quad \frac{\partial T}{\partial t}>0, \quad \frac{\partial^2 T}{\partial x\partial t}>0 \tag{10}$$

For specific $t$, the equilibrium output $x$ must satisfy the two conditions

$$\frac{d\pi_T}{dx}\equiv P(x)+x\frac{dP}{dx}-\frac{dC}{dx}-\frac{\partial T}{\partial x}=0 \tag{11}$$

and

$$\frac{d^2\pi_T}{dx^2}\equiv 2\frac{dP}{dx}+x\frac{d^2P}{dx^2}-\frac{d^2C}{dx^2}-\frac{\partial^2 T}{\partial x^2}<0 \tag{12}$$

(11) defines the monopoly output $x$ as an implicit function of $t$.

In order to evaluate the effect of taxation, we differentiate (11) with respect to $t$:

$$\left(2\frac{dP}{dx}+x\frac{d^2P}{dx^2}-\frac{d^2C}{dx^2}-\frac{\partial^2 T}{\partial x^2}\right)\frac{dx}{dt}-\frac{\partial^2 T}{\partial x\partial t}=0 \tag{13}$$

$(10)_3$, (12) and (13) prove together

$$\frac{dx}{dt}<0 \tag{14}$$

which implies that the output $x$ is a decreasing function of $t$. That is say, both imposition and raise of taxes reduce monopoly output and raise monopoly price.

Let $E(x)$ designate

$$xP(x)-C(x)$$

which is a function of $t$ by (11). Then we have

$$\frac{dE}{dt}=\left(P(x)+x\frac{dP}{dx}-\frac{dC}{dx}\right)\frac{dx}{dt}$$

and it is simplified into

$$\frac{dE}{dt}=\frac{\partial T}{\partial x}\cdot\frac{dx}{dt}$$

by (11). This equality is combined with $(10)_1$ and (14) to prove

$$\frac{dE}{dt}<0 \tag{15}$$

Now, let $x_0$ be the output $x$ for (4), and $x_t$ that for (10) with $t>0$. Then we have

$$x_0P(x_0)-C(x_0)>x_tP(x_t)-C(x_t)$$

since $E(x)$ is a decreasing function of $t$ by (15).

Thus the inequality

$$\{x_0P(x_0)-C(x_0)\}-\{x_tP(x_t)-C(x_t)-T(x_t,t)\}>T(x_t,t)$$

follows easily which implies that the reduction of the net revenue due to taxation is greater than the tax amount itself.

**Cournot Duopoly and Monopoly** In duopoly, the production of the goods is shared between two firms $A$ and $B$. Thus the total output is

$$x=x_1+x_2$$

where $x_1$ is the output of $A$ and $x_2$ that of $B$. The price, however, is left to be determined by the demand function (2) in the same way as in monopoly.

In Cournot duopoly, each duopolist is assumed to act so as to maximize his net revenue provided that the other makes no change in current output no matter how he changes his own.

The net revenue $\pi_1$ of $A$ is

$$\pi_1=\Pi_1(x_1,x_2)\equiv x_1P(x_1+x_2)-C_1(x_1)$$

where $C_1(x_1)$ is the total cost function for $A$. Differentiations with respect to $x_1$ yield the equilibrium and stability conditions

$$\frac{d\pi_1}{dx_1}\equiv P(x_1+x_2)+x_1\frac{dP}{dx_1}-\frac{dC_1}{dx_1}=0 \tag{15}$$

and

$$\frac{d^2\pi_1}{dx_1^2} \equiv 2\frac{dP}{dx_1} + x_1\frac{d^2P}{dx_1^2} - \frac{d^2C_1}{dx_1^2} < 0 \tag{16}$$

for $A$ respectively.

(15) defines $x_1$ as an implicit function of $x_2$ and is geometrically designated as the reaction curve of $A$ on $B$.

Similarly

$$\pi_2 = \Pi_2(x_1, x_2) \equiv x_2 P(x_1 + x_2) - C_2(x_2)$$

is the net revenue of $B$ with the total cost function $C_2(x_2)$. The two conditions for $B$ are

$$\frac{d\pi_2}{dx_2} \equiv P(x_1 + x_2) + x_2\frac{dP}{dx_2} - \frac{dC_2}{dx_2} = 0 \tag{17}$$

and

$$\frac{d^2\pi_2}{dx_2^2} \equiv 2\frac{dP}{dx_2} + x_2\frac{d^2P}{dx_2^2} - \frac{d^2C_2}{dx_2^2} < 0 \tag{18}$$

the former (17) of which is the reaction curve of $B$ on $A$.

In the normal case, the reaction curve is downward sloping with the gradient less than unity: we have

$$-1 < \frac{dx_1}{dx_2} < 0 \tag{19}$$

on (15) and

$$-1 < \frac{dx_2}{dx_1} < 0 \tag{20}$$

on (17).

On (15), $x = x_1 + x_2$ is a function of a single variable $x_2$ by (15) itself, and we have

$$\frac{dx}{dx_2} = \frac{dx_1}{dx_2} + 1$$

which is positive by (19). Therefore $x$ is an increasing function of $x_2$. Accordingly we conclude that the total output for duopoly is greater than that for monopoly and the duopoly price is smaller than the monopoly price.

**Taxation and Duopoly** Suppose that the duopolists are subject to taxes and $T(x, t)$ is the tax amount depending on output $x$ and taxation parameter $t$ with

$$\frac{\partial T}{\partial x} > 0, \qquad \frac{\partial T}{\partial t} > 0, \qquad \frac{\partial^2 T}{\partial x \partial t} > 0.$$

Then their net revenues are

$$\pi_{T1} = \Pi_{T1}(x_1, x_2, t) \equiv x_1 P(x_1 + x_2) - C_1(x_1) - T(x_1, t)$$

and

$$\pi_{T2} = \Pi_{T2}(x_1, x_2, t) \equiv x_2 P(x_1 + x_2) - C_2(x_2) - T(x_2, t)$$

The duopoly outputs $(x_1, x_2)$ must satisfy two systems of conditions:

$$\left\{\begin{aligned} &\frac{d\pi_{T1}}{dx_1} \equiv P(x_1 + x_2) + x_1\frac{dP}{dx_1} - \frac{dC_1}{dx_1} - \frac{\partial T}{\partial x_1} = 0 \\ &\frac{d^2\pi_{T1}}{dx_1^2} \equiv 2\frac{dP}{dx_1} + x_1\frac{d^2P}{dx_1^2} - \frac{d^2C_1}{dx_1^2} - \frac{\partial^2 T}{\partial x_1^2} < 0 \end{aligned}\right. \tag{21}$$

and

$$\left\{\begin{aligned} &\frac{d\pi_{T2}}{dx_2} \equiv P(x_1 + x_2) + x_2\frac{dP}{dx_2} - \frac{dC_2}{dx_2} - \frac{\partial T}{\partial x_2} = 0 \\ &\frac{d^2\pi_{T2}}{dx_2^2} \equiv 2\frac{dP}{dx_2} + x_2\frac{d^2P}{dx_2^2} - \frac{d^2C_1}{dx_2^2} - \frac{\partial^2 T}{\partial x_2^2} < 0. \end{aligned}\right. \tag{22}$$

For the sake of simplicity, $(21)_1$ and $(22)_1$ shall be written hereafter as follows:

$$\begin{cases} F_1(x_1, x_2, t)=0 \\ F_2(x_1, x_2, t)=0 \end{cases} \tag{23}$$

This system of equations defines the duopoly outputs $(x_1, x_2)$ as functions of a single variable $t$ and yields geometrically the reaction curves of the duopolists on each other.

By means of (23), $(21)_2$ and $(22)_1$ can be written as

$$\begin{cases} \dfrac{\partial F_1}{\partial x_1}<0 \\ \dfrac{\partial F_2}{\partial x_2}<0. \end{cases} \tag{24}$$

Differentiation of $(23)_1$ with respect to $x_2$ yield

$$\frac{\partial F_1}{\partial x_1}\frac{dx_1}{dx_2}+\frac{\partial F_1}{\partial x_2}=0$$

i.e.

$$\frac{dx_1}{dx_2}=-\frac{\partial F_1}{\partial x_2}\Big/\frac{\partial F_1}{\partial x_1}$$

which is the direction coefficient of $A$'s reaction curve on $B$. Since $-1<\frac{dx_1}{dx_2}<0$ and $\frac{\partial F_1}{\partial x_1}<0$ by $(24)_1$, we have

$$\frac{\partial F_1}{\partial x_1}<\frac{\partial F_1}{\partial x_2}<0 \tag{25}$$

Similarly

$$\frac{\partial F_2}{\partial x_2}<\frac{\partial F_2}{\partial x_1}<0 \tag{26}$$

follows from $(23)_2$, $-1<\frac{dx_2}{dx_1}<0$ and $(24)_2$.

In order to evaluate the effect of taxation, we differentiate (23) with respect to $t$ to have

$$\begin{cases} \dfrac{\partial F_1}{\partial x_1}\dfrac{dx_1}{dt}+\dfrac{\partial F_1}{\partial x_2}\dfrac{dx_2}{dt}+\dfrac{\partial F_1}{\partial t}=0 \\ \dfrac{\partial F_2}{\partial x_1}\dfrac{dx_1}{dt}+\dfrac{\partial F_2}{\partial x_2}\dfrac{dx_2}{dt}+\dfrac{\partial F_2}{\partial t}=0 \end{cases}$$

Substitution of $-\frac{\partial^2 T}{\partial x\partial t}$ for the third term yields

$$\begin{cases} \dfrac{\partial F_1}{\partial x_1}\dfrac{dx_1}{dt}+\dfrac{\partial F_1}{\partial x_2}\dfrac{dx_2}{dt}=\dfrac{\partial^2 T}{\partial x_1\partial t} \\ \dfrac{\partial F_2}{\partial x_1}\dfrac{dx_1}{dt}+\dfrac{\partial F_2}{\partial x_2}\dfrac{dx_2}{dt}=\dfrac{\partial^2 T}{\partial x_2\partial t} \end{cases}$$

to be solved for $\frac{dx_1}{dt}$ and $\frac{dx_2}{dt}$. Thus we have

$$\begin{cases} \dfrac{dx_1}{dt}=\begin{vmatrix} \dfrac{\partial^2 T}{\partial x_1\partial t} & \dfrac{\partial F_1}{\partial x_2} \\ \dfrac{\partial^2 T}{\partial x_2\partial t} & \dfrac{\partial F_2}{\partial x_2} \end{vmatrix}\Big/\Delta \\ \\ \dfrac{dx_2}{dt}=\begin{vmatrix} \dfrac{\partial F_1}{\partial x_1} & \dfrac{\partial^2 T}{\partial x_1\partial t} \\ \dfrac{\partial F_2}{\partial x_1} & \dfrac{\partial^2 T}{\partial x_2\partial t} \end{vmatrix}\Big/\Delta \end{cases} \tag{27}$$

where

$$\Delta=\begin{vmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{vmatrix} \tag{28}$$

Multiplication of (25) and (26) proves

$$\frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial x_2} > \frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial x_1}$$

whence

$$\Delta>0 \tag{29}$$

follows by (28).

Summation of two equalities of (27) yields

$$\frac{dx}{dt}=\frac{d(x_1+x_2)}{dt}=\begin{vmatrix} \frac{\partial^2 T}{\partial x_1 \partial t} & \frac{\partial F_1}{\partial x_2}-\frac{\partial F_1}{\partial x_1} \\ \frac{\partial^2 T}{\partial x_2 \partial t} & \frac{\partial F_2}{\partial x_2}-\frac{\partial F_2}{\partial x_1} \end{vmatrix} \Big/ \Delta \tag{30}$$

Since $\frac{\partial^2 T}{\partial x_1 \partial t}>0$ and $\frac{\partial F_2}{\partial x_2}-\frac{\partial F_2}{\partial x_1}<0$ by (26), we have

$$\frac{\partial^2 T}{\partial x_1 \partial t}\left(\frac{\partial F_2}{\partial x_2}-\frac{\partial F_2}{\partial x_1}\right)<0 \tag{31}$$

Similarly

$$\frac{\partial^2 T}{\partial x_2 \partial t}\left(\frac{\partial F_1}{\partial x_1}-\frac{\partial F_1}{\partial x_2}\right)<0 \tag{32}$$

since $\frac{\partial^2 T}{\partial x_2 \partial t}>0$ and $\frac{\partial F_1}{\partial x_1}-\frac{\partial F_1}{\partial x_2}<0$ by (25).

Addition of (32) to (31) yields

$$\begin{vmatrix} \frac{\partial^2 T}{\partial x_1 \partial t} & \frac{\partial F_1}{\partial x_2}-\frac{\partial F_1}{\partial x_1} \\ \frac{\partial^2 T}{\partial x_2 \partial t} & \frac{\partial F_2}{\partial x_2}-\frac{\partial F_2}{\partial x_1} \end{vmatrix}<0$$

which is combined with (30) and (29) to prove

$$\frac{dx}{dt}<0.$$

Therefore we conclude that both imposition and raise of taxes reduce total output of duopoly and raise duopoly price.

## III

**Problem of Consumer's Behaviour** In the traditional theory of consumer's behaviour, he is assumed to select that vector of demands for goods which is highest on his preference scale provided that the price vector and his expenditure are given. Mathematically, such a vector $(x_1, x_2, \dots, x_n)$ is required as maximizes

$$u=U(x_1, x_2, \dots, x_n)$$

under the condition

$$p_1x_1+p_2x_2+\cdots+p_nx_n=M$$

where $x_i$ is the demand for the $i$ th goods, $p_i$ its given price, $M$ the consumer's expenditure and $u$ his preference scale.

The fundamental problems are

1. To construct the equilibrium equations for $(x_1, x_2, \dots, x_{n-1}, x_n)$ together with the stability inequalities which define each $x_i$ as a function of $p_1, p_2, \dots, p_n$ and $M$ either implicitly or explicitly,

2. To deduce various equalities and inequalities which might be relevant to the subject and

3. To evaluate various quantities which might be important from comparative statics' point of view.

I think, however, that there is another possible analysis of the problem: the consumer is assumed to act so as to maximize his scale of preference which is possibly different from traditional one under the constraint

$$p_1x_1+p_2x_2+\cdots+p_nx_n \leqq M.$$

It might not be impossible either that he is assumed to act without any restriction on expenditure, i.e. he does not care whether $\sum_i p_ix_i$ is small or large in order to maximize his scale of preference.

Thus let the preference scale of a consumer be

$$u=U(X_1, X_2, \dots, X_n, M) \tag{1}$$

$$\text{with} \quad \frac{\partial U}{\partial M}>0, \tag{2}$$

where $X_i$ is the quantity of the $i$ th goods he possesses and $M$ the amount of his money.

If their current values are $X_1=\alpha_1$, $X_2=\alpha_2, \dots, X_n=\alpha_n$ and $M=\mu$ and the consumer is confronted with given prices $p_1, p_2, \dots, p_n$ of the goods, he will select such a vector $(x_1, x_2, \dots, x_n)$ as will maximize

$$U(\alpha_1+x_1, \alpha_2+x_2, \dots, \alpha_n+x_n, M-\sum_i p_ix_i) \tag{3}$$

**Equilibrium and Stability Conditions** Partial differentiations of (3) with respect to $x_i(i=1,2,\dots,n)$ yield the equilibrium conditions

$$\frac{\partial U}{\partial X_i}-p_i\frac{\partial U}{\partial M}=0 \qquad (i=1,2,\dots,n) \tag{4}$$

to be solved for $(x_1, x_2, \dots, x_n)$, where $\frac{\partial U}{\partial X_i}$ and $\frac{\partial U}{\partial M}$ are partial differential coefficients at $X_j=\alpha_j+x_j$ $(j=1,2,\dots,n)$ and $M=\mu-\sum_i p_ix_i$. This system (4) of equations defines each $x_k$ as a function of $\alpha_1, \alpha_2, \dots, \alpha_n$, $p_1, p_2, \dots, p_n$ and $\mu$ either implicitly or explicitly.

We differentiate the left member of each equation of the system (4) to have

$$\frac{\partial^2 U}{\partial X_j \partial X_i}-p_j\frac{\partial^2 U}{\partial M \partial X_i}-p_i\frac{\partial^2 U}{\partial X_j \partial M}+p_jp_i\frac{\partial^2 U}{\partial M^2} \tag{5}$$

$$(i, j=1,2,\dots,n),$$

where each partial derivative designates the corresponding partial differential coefficient at $X_k=\alpha_k+x_k$ $(k=1,2,\dots,n)$ and $M=\mu-\sum_i p_ix_i$.

Taking (5) as the $i\cdot j$ element of a matrix, the matrix

$$\Delta=\left[\frac{\partial^2 U}{\partial X_j \partial X_i}-p_j\frac{\partial^2 U}{\partial M \partial X_i}-p_i\frac{\partial^2 U}{\partial X_j \partial M}+p_jp_i\frac{\partial^2 U}{\partial M^2}\right]_{i,j} \tag{6}$$

shall be constructed. This must be a matrix of a definite negative quadratic form for maximal scale of preference as opposed to minimal.

(4) is equivalent to

$$\frac{\frac{\partial U}{\partial X_1}}{p_1}=\frac{\frac{\partial U}{\partial X_2}}{p_2}=\cdots=\frac{\frac{\partial U}{\partial X_n}}{p_n}=\frac{\partial U}{\partial M} \tag{*}$$

which is a so-called law of equality and *formally* just similar to the law of equal marginal utility in the traditional theory.

**Evaluation of Comparative Statics' Quantities** In order to evaluate the effects of change of price on demands, each equation of the system (4) shall be partially differentiated with respect to a specific $p_k$. Thus we have

$$\left(\sum_j \frac{\partial^2 U}{\partial X_j \partial X_i}\frac{\partial X_j}{\partial p_k}+\frac{\partial^2 U}{\partial M \partial X_i}\cdot\frac{\partial M}{\partial p_k}\right)$$
$$-\delta_{ik}\frac{\partial U}{\partial M}-p_i\left(\sum_j \frac{\partial^2 U}{\partial X_j \partial M}\cdot\frac{\partial X_j}{\partial p_k}+\frac{\partial^2 U}{\partial M^2}\frac{\partial M}{\partial p_k}\right)=0$$
$$i=1,2,\dots,n,$$

which becomes

$$\sum_j\left(\frac{\partial^2 U}{\partial X_j\partial X_i}-p_j\frac{\partial^2 U}{\partial M\partial X_i}-p_i\frac{\partial^2 U}{\partial X_j \partial M}+p_jp_i\frac{\partial^2 U}{\partial M^2}\right)\frac{\partial x_j}{\partial p_k}$$
$$=\delta_{ik}\frac{\partial U}{\partial M}+x_k\left(\frac{\partial^2 U}{\partial M\partial X_i}-p_i\frac{\partial^2 U}{\partial M^2}\right) \tag{7}$$
$$i=1,2,\dots,n$$

by substitution of $\frac{\partial x_j}{\partial p_k}$ for $\frac{\partial X_j}{\partial p_k}$ and that of $-x_k-\sum_j p_j\frac{\partial x_j}{\partial p_k}$ for $\frac{\partial M}{\partial p_k}$.

On solving (7) for $\frac{\partial x_j}{\partial p_k}$, we have

$$\frac{\partial x_j}{\partial p_k}=\frac{D_{kj}}{D}\cdot\frac{\partial U}{\partial M}+x_k\frac{\sum_i\left(\frac{\partial^2 U}{\partial M\partial X_i}-p_i\frac{\partial^2 U}{\partial M^2}\right)D_{ij}}{D} \tag{8}$$
$$j=1,2,\dots,n,$$

where $D$ is the determinant of the matrix $\Delta$ and $D_{st}$ is its $i\cdot j$ cofactor.

This is the effect of change of price of the $k$ th goods on demand of the $j$-th goods.

In order to evaluate the effects of change of money amount on demands, each equation of the system (4) shall be partially differentiated with respect to $\mu$. Thus we have

$$\left(\sum_j \frac{\partial^2 U}{\partial X_j \partial X_i}\frac{\partial X_j}{\partial \mu}+\frac{\partial^2 U}{\partial M \partial X_i}\frac{\partial M}{\partial \mu}\right)$$
$$-p_i\left(\sum_j \frac{\partial^2 U}{\partial X_j \partial M}\frac{\partial X_j}{\partial \mu}+\frac{\partial^2 U}{\partial M^2}\frac{\partial M}{\partial \mu}\right)=0$$
$$i=1,2,\dots,n,$$

which becomes

$$\sum_j\left(\frac{\partial^2 U}{\partial X_j\partial X_i}-p_j\frac{\partial^2 U}{\partial M\partial X_i}-p_i\frac{\partial^2 U}{\partial X_j \partial M}+p_jp_i\frac{\partial^2 U}{\partial M^2}\right)\frac{\partial x_j}{\partial \mu}$$
$$=p_i\frac{\partial^2 U}{\partial M^2}-\frac{\partial^2 U}{\partial M\partial x_i} \tag{9}$$
$$i=1,2,\dots,n,$$

by replacement of $\frac{\partial X_j}{\partial \mu}$ by $\frac{\partial x_j}{\partial \mu}$ and that of $\frac{\partial M}{\partial \mu}$ by $1-\sum_j p_j\frac{\partial x_j}{\partial \mu}$.

On solving (9) for $\frac{\partial x_j}{\partial \mu}$, we have

$$\frac{\partial x_j}{\partial \mu}=-\frac{\sum_i\left(\frac{\partial^2 U}{\partial M \partial X_i}-p_i\frac{\partial^2 U}{\partial M^2}\right)D_{ij}}{D} \qquad (10)$$

$$j=1, 2, \ldots, n,$$

which is the effect of change of money amount on demands.

In order to evaluate the effects of change of goods quantity on demand, each equation of the system (4) shall be partially differentiated with respect to a specific $\alpha_k$ to yield

$$\left(\sum_j \frac{\partial^2 U}{\partial X_j \partial X_i}\frac{\partial X_j}{\partial \alpha_k}+\frac{\partial^2 U}{\partial M \partial X_i}\frac{\partial M}{\partial \alpha_k}\right)$$

$$-p_i\left(\sum_j \frac{\partial^2 U}{\partial X_j \partial M}\frac{\partial X_j}{\partial \alpha_k}+\frac{\partial^2 U}{\partial M^2}\frac{\partial M}{\partial \alpha_k}\right)=0$$

$$i=1, 2, \ldots, n.$$

Substitution of $\delta_{jk}+\frac{\partial x_j}{\partial \alpha_k}$ for $\frac{\partial X_j}{\partial \alpha_k}$ and that of $-\sum_j p_j\frac{\partial x_j}{\partial \alpha_k}$ for $\frac{\partial M}{\partial \alpha_k}$ prove

$$\sum_j\left(\frac{\partial^2 U}{\partial X_j \partial X_i}-p_j\frac{\partial^2 U}{\partial X_i \partial M}-p_i\frac{\partial^2 U}{\partial X_j \partial M}+p_jp_i\frac{\partial^2 U}{\partial M^2}\right)\frac{\partial x_j}{\partial \alpha_k}$$

$$=-\sum_j \delta_{jk}\left(\frac{\partial^2 U}{\partial X_j \partial X_i}-p_i\frac{\partial^2 U}{\partial X_j \partial M}\right) \qquad (11)$$

$$i=1, 2, \ldots, n.$$

On solving (11) for $\frac{\partial x_j}{\partial \alpha_k}$, we have

$$\frac{\partial x_j}{\partial \alpha_k}=-\frac{\sum_i\left(\frac{\partial^2 U}{\partial X_k \partial X_i}-p_i\frac{\partial^2 U}{\partial X_k \partial M}\right)D_{ij}}{D} \qquad (12)$$

$$j=1, 2, \ldots, n,$$

which is the effect of change of quantity of the $k$ th goods on demand of the $j$ th goods and is not considered in the traditional theory.

**Mutual Relations of the Quantities Evaluated** The next problem is to reveal relations of these quantities $\frac{\partial x_j}{\partial p_k}$, $\frac{\partial x_j}{\partial \mu}$ and $\frac{\partial x_j}{\partial \alpha_k}$ of comparative statics.

(8) is simplified into

$$\frac{\partial x_j}{\partial p_k}=\frac{D_{kj}}{D}\frac{\partial U}{\partial M}-x_k\frac{\partial x_j}{\partial \mu} \qquad (13)$$

$$j, k=1, 2, \ldots, n$$

by means of (10), which is a formal analogue of the so-called Slutsky's Equality.

(13) combined with symmetricity of $\varDelta$ yields

$$\frac{\partial x_j}{\partial p_k}+x_k\frac{\partial x_j}{\partial \mu}=\frac{\partial x_k}{\partial p_j}+x_j\frac{\partial x_k}{\partial \mu} \qquad (14)$$

$$j, k=1, 2, \ldots, n$$

which is a formal analogue of the so-called Slutsky-Johnson's Equality.

For a specific $j$, the equalities of (13) shall be multiplied by $p_k(k=1, 2, \ldots, n)$ respectively and summed up over $k$. Then we have

$$\sum_k p_k \frac{\partial x_j}{\partial p_k} + \mu \frac{\partial x_j}{\partial \mu} = \frac{\sum_k p_k D_{kj}}{D} \frac{\partial U}{\partial M} + (\mu - \sum_k p_k x_k) \frac{\partial x_j}{\partial \mu}$$

$$j=1, 2, \ldots, n$$

which does not necessarily endorse the 0 degree homogeneity of demand function with respect to prices and money amount though the traditional theory does.

For $k=j$, (13) becomes

$$\frac{\partial x_j}{\partial p_j} + x_j \frac{\partial x_j}{\partial \mu} = \frac{D_{jj}}{D} \frac{\partial U}{\partial M} \tag{15}$$

Since $\Delta$ is the matrix of a definite negative quadratic form

$$\frac{D_{jj}}{D} < 0 \tag{16}$$

By (2), (15) and (16), we have

$$\frac{\partial x_j}{\partial p_j} + x_j \frac{\partial x_j}{\partial \mu} < 0 \tag{17}$$

$$j=1, 2, \ldots, n.$$

This is a formal analogue of the Slutsky-Johnson's Inequality and shows that

$$\frac{\partial x_j}{\partial p_j} < 0$$

does not always follows and that

$$\frac{\partial x_j}{\partial p_j} > 0 \quad \text{and} \quad \frac{\partial x_j}{\partial \mu} > 0$$

do not simultaneously hold.

Let the $s \cdot t$ element of $\Delta$ be designated by $d_{st}$, then

$$\Delta \left[ \left( \frac{\partial x_j}{\partial p_k} + x_k \frac{\partial x_j}{\partial \mu} \right) \Big/ \frac{\partial U}{\partial M} \right]_{jk} = [d_{ij}]_{ij} [D_{jk}/D]_{jk}$$

$$= [\sum_j d_{ij} D_{jk}/D]_{ik} = [\delta_{ik}]_{ik} = E \qquad \text{by (13).}$$

Thus

$$\left[ \left( \frac{\partial x_j}{\partial p_k} + x_k \frac{\partial x_j}{\partial \mu} \right) \Big/ \frac{\partial U}{\partial M} \right]_{jk} = \Delta^{-1}.$$

Since $\Delta$ is the matrix of a definite negative quadratic quadratic form, so is

$$\left[ \left( \frac{\partial x_j}{\partial p_k} + x_k \frac{\partial x_j}{\partial \mu} \right) \Big/ \frac{\partial U}{\partial M} \right]_{jk}.$$

From this and (2), we conclude that

$$\left[ \frac{\partial x_j}{\partial p_k} + x_k \frac{\partial x_j}{\partial \mu} \right]_{jk}$$

is the matrix of a definite negative quadratic form. This is a generalization of (17).

(10) is combined with (12) to yield

$$\frac{\partial x_j}{\partial \alpha_k} - p_k \frac{\partial x_j}{\partial \mu}$$

$$= -\sum_i \left( \frac{\partial^2 U}{\partial X_i \partial X_k} - p_k \frac{\partial^2 U}{\partial M \partial X_i} - p_i \frac{\partial^2 U}{\partial X_k \partial M} + p_k p_i \frac{\partial^2 U}{\partial M^2} \right) D_{ij}/D$$

$$= -\sum_i d_{ik} D_{ij}/D = -\delta_{kj}$$

i.e.

$$\frac{\partial x_j}{\partial \alpha_k} + \delta_{kj} = p_k \frac{\partial x_j}{\partial \mu} \tag{19}$$

$$j, k=1, 2, \ldots, n.$$

The $\dfrac{\partial x_j}{\partial \alpha_k}$s were not taken into consideration in the traditional theory of consumer's behaviour, but they are very simply related to $\dfrac{\partial x_j}{\partial \mu}$ through $p_k$ as were shown in (19).

For each $j$, (19) is rewritten as

$$\frac{\dfrac{\partial x_j}{\partial \alpha_1}}{p_1}=\cdots\cdots=\frac{\dfrac{\partial x_j}{\partial \alpha_{j-1}}}{p_{j-1}}=\frac{\dfrac{\partial x_j}{\partial \alpha_j}+1}{p_j}$$
$$=\frac{\dfrac{\partial x_j}{\partial \alpha_{j+1}}}{p_{j+1}}=\cdots\cdots=\frac{\dfrac{\partial x_j}{\partial \alpha_n}}{p_n}$$

i.e. $$\frac{\dfrac{Ex_j}{E\alpha_1}}{\alpha_1 p_1}=\cdots\cdots=\frac{\dfrac{Ex_j}{E\alpha_{j-1}}}{\alpha_{j-1}p_{j-1}}=\frac{\dfrac{Ex_j}{E\alpha_j}+\dfrac{\alpha_j}{x_j}}{\alpha_j p_j}$$
$$=\frac{\dfrac{Ex_j}{E\alpha_{j+1}}}{\alpha_{j+1}p_{j+1}}=\cdots\cdots=\frac{\dfrac{Ex_j}{E\alpha_n}}{\alpha_n p_n}$$

in terms of elasticity.

This is another law of equality besides (*) in this model of consumer's behaviour.

In order to examine the relation between price effect and quantity effect upon demands, $\dfrac{\partial x_j}{\partial \mu}$ shall be eliminated between (13) and (19).

Thus we have

$$\frac{\partial x_j}{\partial p_k}=\frac{D_{kj}}{D}\frac{\partial U}{\partial M}-\frac{x_k}{p_k}\left(\frac{\partial x_j}{\partial \alpha_k}+\delta_{kj}\right) \qquad (20)$$
$$k, j=1, 2, \ldots, n.$$

For each pair $(k, j)$ of $k$ and $j$, elimination of $\dfrac{\partial U}{\partial M}$ among (20) by means of symmetricity of $\Delta$ yields

$$\frac{\partial x_j}{\partial p_k}+\frac{x_k}{p_k}\left(\frac{\partial x_j}{\partial \alpha_k}+\delta_{kj}\right)=\frac{\partial x_k}{\partial p_j}+\frac{x_j}{p_j}\left(\frac{\partial x_k}{\partial \alpha_j}+\delta_{jk}\right)$$
$$k, j=1, 2, \ldots, n.$$

For $k=j$, they are trivial, but for $k\neq j$ we have

$$\frac{\partial x_j}{\partial p_k}+\frac{x_k}{p_k}\frac{\partial x_j}{\partial \alpha_k}=\frac{\partial x_k}{\partial p_j}+\frac{x_j}{p_j}\frac{\partial x_k}{\partial \alpha_j}$$
$$k, j=1, 2, \ldots, n \quad \text{and} \quad k\neq j.$$

We combine (17) with (19) to have

$$\frac{\partial x_j}{\partial p_j}+\frac{x_j}{p_j}\left(\frac{\partial x_j}{\partial \alpha_j}+1\right)<0$$

i.e.

$$\frac{Ex_j}{Ep_j}+\frac{\partial x_j}{\partial \alpha_j}+1<0$$
$$j=1, 2, \cdots, n$$

in terms of elasticity.

## IV

The problem of monopoly and duopoly is one of my pet examples. It employs almost all techniques of differential calculus, introduces tax amount in an extremely general form and furnishes us with an easily understandable problem of comparative statics.

The above-mentioned model of consumer's behaviour might be more realistic than the traditional. The latter specifies the consumer's expenditure strictly. He should, however, select a vector of goods quantities which is the highest on his preference scale irrespective of the preassigned budget.

(1965•10•8)